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Semiclassical investigation of a charged relativistic membrane model

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Abstract. A semiclassical and variational estimate of the energy of a particular type of charged membrane is studied and results are found at variance with a previous harmonic estimate. We briefly comment on possible alternatives.

1. Introduction

In a recent note [1] we introduced a relativistic membrane model based on an extremum of the action

$$\mathscr{A}[C]=\kappa' \int_C H \hat{*} 1+\tfrac{1}{2}\int_{\mathcal{M}} F\wedge *F \tag{1}$$

where H is the magnitude of the trace of the shape tensor associated with a three-dimensional timelike immersion C into spacetime M. The membrane was electrically charged and coupled to the Maxwell field F in a manner first discussed by Dirac [2]. The map C is determined from the equation of motion

$$\kappa\hat{\mathscr{R}}=\tfrac{1}{2}*^{-1}(F\wedge *F)|_\Sigma \tag{2}$$

where Σ is the image of C, $\kappa=-\kappa'/3$, $\hat{\mathscr{R}}$ is the Gaussian curvature of Σ and $d * F=0$ on $\mathcal{M}\subset M$ with $\partial\mathcal{M}=\Sigma$. In these formulae $*$ is the Hodge map of the Minkowski spacetime metric and $\hat{*}$ refers to the induced metric on Σ. We observed in [1] that for certain spherically symmetric configurations (where Σ has the topology $S^2\times \mathbf{R}$) small radial oscillations about equilibrium exist with angular frequency $\omega=m_1/e^2$ where m_1 is the equilibrium rest mass and e is the charge on the membrane in geometrised units. The energy of a harmonic quantum associated with these oscillations suggested that an attempt be made to find a quantum model for the lepton spectrum based on the action in equation (1). In this paper we report on our efforts in this direction.

2. Bohr–Sommerfeld estimate

In standard spherical polar coordinates $\{t, r, \theta, \phi\}$ about any point in Minkowski spacetime we consider the immersion

$$C:[\tau,\sigma,\rho]\mapsto(t=\tau,\ r=R(\tau),\ \theta=\sigma,\ \phi=\rho) \tag{3}$$

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for some function R. In this chart the equation of motion (2) reads

$$\kappa\left(\frac{4\ddot{R}}{R(1-\dot{R}^2)^2}+\frac{2}{R^2(1-\dot{R}^2)}\right)=\frac{e^2}{2R^4}. \tag{4}$$

This can be derived from the effective Lagrangian

$$L=2\kappa R\dot{R}\tanh^{-1}\dot{R}-V(R) \tag{5}$$

with

$$V(R)=2\kappa R+\frac{e^2}{2R}.$$

Equation (5) differs from the effective Lagrangian given in [1] by a total time derivative. The momentum conjugate to R is

$$p=2\kappa R\left(\tanh^{-1}\dot{R}+\frac{\dot{R}}{1-\dot{R}^2}\right). \tag{6}$$

If we introduce a variable $\Gamma=(1-\dot{R}^2)^{-1}$ then the canonical Hamiltonian is

$$H(p,R)=p\dot{R}-L=2\kappa R\Gamma(p,R)+e^2/2R \tag{7}$$

where Γ is a solution of

$$\tanh\left(\frac{p}{2\kappa R}-[\Gamma(\Gamma-1)]^{1/2}\right)=\left(\frac{\Gamma-1}{\Gamma}\right)^{1/2}. \tag{8}$$

In terms of the energy

$$E=\frac{2\kappa R}{1-\dot{R}^2}+\frac{e^2}{2R} \tag{9}$$

we have

$$p=\left[\left(E-\frac{e^2}{2R}\right)\left(E-2\kappa R-\frac{e^2}{2R}\right)\right]^{1/2}+2\kappa R\tanh^{-1}\left(\frac{E-2\kappa R-e^2/2R}{E-e^2/2R}\right)^{1/2}. \tag{10}$$

The turning points of the classical motion with fixed energy $E>m_1$ are the roots $R_\pm$ of the quadratic equation

$$E=2\kappa R_\pm+\frac{e^2}{2R_\pm} \tag{11}$$

and so we may alternatively write

$$p=\frac{1}{R}(2\kappa E)^{1/2}[(R-r)(R_+-R)(R-R_-)]^{1/2}$$
$$+2\kappa R\tanh^{-1}\left[\left(\frac{2\kappa}{E}\right)^{1/2}\left(\frac{(R_+-R)(R-R_-)}{(R-r)}\right)^{1/2}\right] \tag{12}$$

where $r=R_+R_-/(R_++R_-)$. According to the (modified) semiclassical Bohr-Sommerfeld quantisation condition [3] for periodic systems,

$$\int_{R_-}^{R_+}p\,\mathrm{d}R=\pi\hbar(n+\tfrac{3}{4}) \tag{13}$$

for some integers $n=0,1,2,\ldots$. Inserting (12) into (13) and integrating by parts we obtain

$$\int_{R_-}^{R_+} p\,\mathrm{d}R=(2\kappa E)^{1/2}\bigg(\int_{R_-}^{R_+}[(R-r)(R_+-R)(R-R_-)]^{1/2}\frac{\mathrm{d}R}{R}$$
$$+\tfrac{1}{4}\int_{R_-}^{R_+}\frac{R(R-2r)}{[(R-r)(R_+-R)(R-R_-)]^{1/2}}\,\mathrm{d}R\bigg)=\pi\hbar(n+\tfrac{3}{4}). \quad (14)$$

This condition may be expressed in terms of the dimensionless variables

$$\mu=E/m_1 \quad (15)$$

$$\rho_\pm(\mu)=1\pm\left(1-\frac{1}{\mu^2}\right)^{1/2} \quad (16)$$

$$k(\mu)=\left[1-\left(\frac{\rho_-}{\rho_+}\right)^2\right]^{1/2} \quad (17)$$

$$\alpha=e^2/\hbar=\tfrac{1}{137} \qquad \text{and} \qquad \beta=1-\rho_-/\rho_+ \quad (18)$$

and the elliptic integrals [4]

$$\mathbf{F}(k)=\int_0^1\frac{\mathrm{d}x}{[(1-x^2)(1-k^2x^2)]^{1/2}} \quad (19)$$

$$\mathbf{E}(k)=\int_0^1\left(\frac{1-k^2x^2}{1-x^2}\right)^{1/2}\mathrm{d}x \quad (20)$$

$$\Pi(\beta,k)=\int_0^1\frac{\mathrm{d}x}{(1-\beta x^2)[(1-x^2)(1-k^2x^2)]^{1/2}} \quad (21)$$

as

$$\mu^2(\rho_+\mathbf{E}(k)+\rho_-^2\Pi(\beta,k)-2\rho_-\mathbf{F}(k))=\frac{\pi}{\alpha}(n+\tfrac{3}{4}). \quad (22)$$

We have computed the left-hand side of (22) numerically as a function of μ and compared it with the right-hand side for various values of n. For $n=0$, $\mu\simeq 13$ and one needs $n\simeq 190$ to get $\mu\simeq 200$. (It is worth noting that the ratio of the ground-state energy based on the Bohr-Sommerfeld estimate to the harmonic result is comparable in this model to the same ratio of estimates found by Dirac in his model).

3. Variational estimate

How reliable is the Bohr-Sommerfeld method for the type of Hamiltonian under consideration here? A proper answer must require a comparison with a more exact treatment. Clearly the p, R canonical variables are not very suitable for our purpose in this respect. However, in searching for an alternative canonical representation of our classical Hamiltonian one must be aware that different ordering assignments of the canonical variables will in general define different quantum theories.

Consider the canonical transformation $p, R\mapsto P, Q$ where

$$P=\frac{p}{2\kappa R}-[\Gamma(\Gamma-1)]^{1/2} \quad (23)$$

$$Q=2\kappa R^2\Gamma. \quad (24)$$

We readily find that

$$p\,\mathrm{d}R - H(p,R)\,\mathrm{d}t = P\,\mathrm{d}Q - \sqrt{2\kappa}\left(\sqrt{Q} + \frac{e^2}{2\sqrt{Q}}\right)\cosh P\,\mathrm{d}t \tag{25}$$

(mod d) so that P and Q are indeed canonical. In terms of the dimensionless variables $q = Q/\hbar$ and $\mathcal{H} = H/m_1$ we have the Hamiltonian

$$\mathcal{H}(P,q) = \frac{1}{\sqrt{2\alpha}}\left(\sqrt{q} + \frac{\alpha}{2\sqrt{q}}\right)\cosh P \tag{26}$$

in a product form. The classical equilibrium configuration is given by $P = 0$ and $q = \frac{1}{2}\alpha$, when $\mathcal{H} = 1$. Hence we define

$$\chi = q - \tfrac{1}{2}\alpha \tag{27}$$

$$\mathcal{H}' = \mathcal{H} - 1 \tag{28}$$

so that

$$\mathcal{H}' = \tfrac{1}{2}\left[\left(1 + \frac{2\chi}{\alpha}\right)^{1/2} + \left(1 + \frac{2\chi}{\alpha}\right)^{-1/2}\right]\cosh P - 1. \tag{29}$$

In this form by expanding $\cosh P$ we see that if $2\chi/\alpha \ll 1$

$$\mathcal{H}' = \tfrac{1}{2}(P^2 + \chi^2/\alpha^2) - (\chi/\alpha)^3 + \tfrac{1}{24}P^4 + \ldots. \tag{30}$$

Although we recover the classical harmonic limit from this expansion for suitably small χ we see that the higher-order terms are not perturbative in α. At a more fundamental level we note that there is no obvious choice of ordering the canonical variables to define any particular operator Hamiltonian for a quantum description.

To gain some further insight into the quantum theory we have examined (7) in the limit of small radial velocities. In this situation we have defined the theory by adopting for a quantum Hamiltonian

$$\hat{H} = \frac{1}{16\kappa}\left(\frac{1}{R}\hat{p}^2 + \hat{p}^2\frac{1}{R}\right) + 2\kappa R + \frac{e^2}{2R} \tag{31}$$

in a Hilbert space equipped with the inner product

$$(\phi, \psi) = 4\pi\int_0^\infty \phi^*(R)\psi(R)R^2\,\mathrm{d}R. \tag{32}$$

Representing $\hat{p}^2$ in this space by the Hermitian operator

$$-\frac{\hbar^2}{R}\frac{\partial^2}{\partial R^2}(R\quad)$$

we have performed a variational estimate of the ground-state energy based on a one-parameter trial function

$$\psi_\lambda(\rho) = \rho\exp(-\tfrac{1}{2}\lambda\rho^2) \tag{33}$$

where $R = R_0\rho$ in terms of the classical equilibrium radius $R_0 = e^2/m_1$. In terms of λ we find

$$\frac{3\sqrt{\pi}}{2}\frac{E(\lambda)}{m_1} = \frac{2}{\sqrt{\lambda}} + \sqrt{\lambda} + \frac{\lambda^{3/2}}{\alpha^2} \tag{34}$$

which is minimised when

$$\lambda = \tfrac{1}{6}[\alpha(\alpha^2+24)^{1/2} - \alpha^2]. \tag{35}$$

Thus if the variational theorem is applicable to the Hamiltonian (31) then the associated ground-state energy turns out to be less than $13m_1$.

4. Generalisations

Finally we comment on a more general (two-parameter) model in which Dirac's original action is augmented with the action depending on the shape tensor:

$$\mathscr{A}[C] = \kappa_1' \int_C \hat{*}1 + \kappa_2' \int_C H \hat{*} 1 + \tfrac{1}{2}\int_{\mathcal{M}} F \wedge *F. \tag{36}$$

In this situation the effective Lagrangian for spherical configurations (3) is

$$L = -\kappa_1 R^2(1-\dot{R}^2)^{1/2} + 2\kappa_2 R\dot{R}\tanh^{-1}\dot{R} - 2\kappa_2 R - e^2/2R. \tag{37}$$

The equation of motion for $R(t)$

$$\frac{\kappa_1}{R^2\dot{R}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{R^2}{(1-\dot{R}^2)^{1/2}}\right) + 2\kappa_2\left[\frac{2}{R(1-\dot{R}^2)^{1/2}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{R}}{(1-\dot{R}^2)^{1/2}}\right) + \frac{1}{R^2(1-\dot{R}^2)}\right] = \frac{e^2}{2R^4} \tag{38}$$

has the first integral

$$E = \frac{\kappa_1 R^2}{(1-\dot{R}^2)^{1/2}} + \frac{2\kappa_2 R}{1-\dot{R}^2} + \frac{e^2}{2R}. \tag{39}$$

The equilibrium configuration $R = R_0$ with energy $E = m_1$ relates the constants κ_1, κ_2 to R_0 and m_1

$$\kappa_1 + \frac{\kappa_2}{R_0} - \frac{e^2}{4R_0^3} = 0 \tag{40}$$

$$\kappa_1 R_0^2 + 2\kappa_2 R_0 + \frac{e^2}{2R_0} = m_1. \tag{41}$$

As before the canonical Hamiltonian is given implicitly in terms of p and R by

$$H(p, R) = \kappa_1 R^2 \Gamma^{1/2} + 2\kappa_2 R\Gamma + e^2/2R \tag{42}$$

where Γ in this case is a solution of

$$p = \kappa_1 R^2(\Gamma-1)^{1/2} + 2\kappa_2 R\left[[\Gamma(\Gamma-1)]^{1/2} + \tanh^{-1}\left(\frac{\Gamma-1}{\Gamma}\right)^{1/2}\right]. \tag{43}$$

The canonical transformation $(p, R) \mapsto (P, Q)$

$$P = \frac{p - \kappa_1 R^2(\Gamma-1)^{1/2}}{2\kappa_2 R} - [\Gamma(\Gamma-1)]^{1/2} \tag{44}$$

$$Q = \tfrac{1}{3}\kappa_1 R^3 \Gamma^{1/2} + 2\kappa_2 R^2 \Gamma \tag{45}$$

gives

$$H(P, Q) = \kappa_1 R^2 \cosh P + 2\kappa_2 R \cosh^2 P + e^2/2R \tag{46}$$

where $R(P, Q)$ is a solution of the cubic equation

$$\tfrac{1}{3}(\kappa_1 \cosh P)R^3 + (2\kappa_2 \cosh^2 P)R^2 - Q = 0. \qquad (47)$$

We do not pursue this model further here.

5. Conclusions

The semiclassical Bohr–Sommerfeld and variational approaches used to estimate the energy of radial modes in our model give results that are an order of magnitude lower than the estimate based on the energy of a harmonic quantum. We remarked in [1] that even at the classical level one could not regard an oscillation with $E/m_1 \simeq 200$ to have a period independent of amplitude. Although the above estimates are not conducive to the idea that our model describe a spectrum of excited lepton states they do not in themselves rule out the possibility that such a spectrum may have its origin in some extended particle model. The Bohr–Sommerfeld approach by its very nature makes no explicit reference to the effects of operator ordering. Ordering ambiguities are clearly evident in the canonical representation of our Hamiltonian. In the variational approach we have of necessity had to neglect higher-order operators in the expansion of the Hamiltonian and we have given reasons to suggest that such terms cannot be ignored.

There may be other choices of canonical variables that are more appropriate for a passage to the quantum theory. In a complete treatment one would expect to represent the algebra of the Lorentz group on the space of quantum states for the membrane. It is not difficult to find new canonical variables in which the Hamiltonian no longer depends on a hyperbolic function. However, the appealing product form of (26) is then lost.

We have discussed in this paper a charged membrane model in terms of certain geometrical properties of a three-dimensional timelike immersion into spacetime. The actions discussed here are the simplest that give rise to classical differential equations of motion no higher than second order. The model based on [1] gives a surprising estimate for the rest mass of an excited state in the quantised harmonic approximation; a result that is not consistent with a semiclassical Bohr–Sommerfeld estimate. However, it is not immediately clear that such an estimate is completely reliable for our model since we have not found a representation of the Hamiltonian offering a natural ordering in which to formulate an exact quantum theory. Whilst the harmonic estimate must remain an unreliable one the question of a proper quantisation of our model remains open.

We have briefly examined a model in which both intrinsic and extrinsic curvatures play a role in the dynamics to see if any simplifications occur. From our preliminary analysis this seems unlikely. It may be that in models of this type the neglect of Casimir effects and fermionic properties is unwarranted.

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